Integral Relations Method Computations of Annular and Asymmetric Plane Nozzle Flowfields

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The method of integral relations has been used to compute the flowfield in axisymmetric nozzles. In this paper, the method is extended to include annular nozzles and asymmetric plane nozzles such as used in turbine blading. The results of the flowfield calculations for the annular nozzle are compared to experimentally obtained wall pressure measurements. The results for the turbine nozzle blade are compared to results obtained using the Katsanis method. Both compressible and incompressible flows are considered.

Introduction

THE method of integral relations was originated by A. A. Dorodnitsyn[1] in 1958 and was subsequently applied to a number of problems in fluid dynamics. The application of the method to nozzle flow has been made by Holt,[2] by Alikhashkin, Favorskii and Chushkin,[3] and by Van Zhu-Tsuan.[4] In all of these papers, the shape of the nozzle wall was determined for a given centerline velocity or velocity potential distribution (the indirect problem). The first application of the method to the direct problem (determining the flowfield for a given nozzle shape) was made by Liddle and Archer[5] in 1968 for an axisymmetric nozzle without a centerbody. Two additional papers were published in 1971.[6,7] These papers were limited to nozzles where the abscissa was the axis of symmetry and was contained in the flowfield.

Nomenclature

r, z = cylindrical or Cartesian coordinate (Fig. 1)
u, w = radial and axial velocity components (Fig. 1)
i = strip boundary index
j = number of strips
n = coordinate conversion factor (zero for Cartesian, one for cylindrical)
P, P_{stag} = pressure and stagnation pressure, respectively
V = total velocity
V_{max} = maximum velocity (Eq. 1)
ρ = density
γ = specific heat ratio

In this paper, the method is extended to annular and asymmetric plane nozzles where the abscissa is no longer in the field and may, in fact, be located a considerable distance away. The method is amenable to both compressible and incompressible flows and both are considered here. The flow in both cases is assumed to be inviscid, adiabatic, and irrotational. In the compressible flow case, the specific heat ratio of the fluid is considered constant and the flow is assumed to be free of shocks.

The coordinate system used is shown in Fig. 1. The coordinates are referred to as r and z for both the annular nozzle (cylindrical coordinates) and the asymmetric plane nozzle (Cartesian coordinates). In applying the method of integral relations, the flowfield is divided into strips evenly spaced between the walls, as shown in Fig. 1. To generalize the discussion for an arbitrary number of strips j, the values of parameters at the strip boundaries are designated by a double subscript, wherein the first subscript i defines the particular strip boundary in the field of j strips. Thus the inner or lower wall is always designated by $i = 0$ and the outer or upper wall by $i = j$. Intermediate lines have integer values of i between 0 and j counted outward from the inner wall.

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Basic Equations

The basic equations of motion for incompressible flow, the continuity, irrotationality, and Bernoulli's equation, are rendered dimensionless by referring the velocities to the maximum velocity defined by Eq. (1), the pressures to their stagnation values and the lengths either to the radius of the outer wall at $z = 0$ in the case of the annular nozzle, or to the channel width at $z = 0$ in the case of the asymmetric plane nozzle. The maximum velocity is defined as

$$V_{max} = \left(\frac{2P_{stag}}{\rho}\right)^{1/2} \tag{1}$$

The forms of the continuity and irrotationality equations are not changed by normalization and from this point on, the

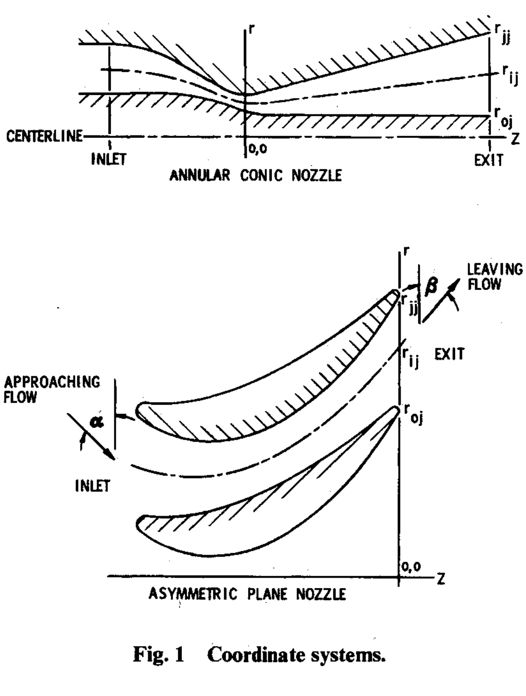

Fig. 1 Coordinate systems.

symbols are taken to indicate normalized values. Bernoulli's equation, however, becomes

$$P+V^2=1 \tag{2}$$

where, again, the symbols are indicative of normalized values.

The compressible flow equations for continuity and irrotationality are not changed by normalization. The normalized equations for entropy and momentum are

Entropy

$$P=\rho^{\gamma} \tag{3}$$

and Integrated Momentum

$$\rho=\{1-[(\gamma-1)/(\gamma+1)]V^2\}^{1/(\gamma-1)} \tag{4}$$

The pressure and density have been normalized by referring them to their respective stagnation values. The velocity has been normalized by referring it to the critical velocity, C^*, given by Eq. (5).

$$C^*=\left[\frac{2\gamma P_{\text{stag}}}{(\gamma+1)\rho_{\text{stag}}}\right]^{1/2} \tag{5}$$

The reference length is the same as for incompressible flow, that is, either the radius of the outer wall or the channel width at $z=0$.

In both flow cases, the boundary conditions on velocity are

outer wall

$$u_{jj}=w_{jj}(dr_{jj}/dz) \tag{6}$$

inner wall

$$u_{0j}=w_{0j}(dr_{0j}/dz) \tag{7}$$

Application of the Method

The basic equations for continuity and irrotationality are in the desired "divergence" form required by the method of integral relations.[1] The use of the method has been illustrated for ordinary nozzles[5–7] and a full derivation of the equations for a number of cases has been presented.[8]

The method is illustrated here using the irrotationality equation for the case when the inner wall radius $r=r_{0j}$ is used instead of the $r=0$ used for ordinary nozzles. The flowfield is divided into j strips. The irrotationality equation, when integrated across the strips in the r direction, yields j ordinary differential equations if the remaining integrands are approximated. The approximations are called the "integral relations" and use the values of the integrands at the strip boundaries. As a result, the order of the approximations depends on the number of strips.

In the simple one strip solution, the approximation employs only values of the integrands at the inner and outer walls. Integrating the irrotationality equations from the inner wall, $r=r_{01}$ to the outer wall, r_{11} yields

$$w_{11}-w_{01}-\int_{r_{01}}^{r_{11}}\frac{\partial u}{\partial z}\,dr=0 \tag{8}$$

Applying Liebnitz's Rule

$$w_{11}-w_{01}-\frac{d}{dz}\int_{r_{01}}^{r_{11}}u\,dr+u_{11}\frac{dr_{11}}{dz}-u_{01}\frac{dr_{01}}{dz}=0 \tag{9}$$

An approximation must now be made for the integrand u. It is presumed to be a continuous and bounded function of $(r-r_{0j})/(r_{jj}-r_{0j})$ and must, by symmetry, be an odd-powered function. In the case of the asymmetric plane nozzle, it can be assumed that there is a similar channel below the axis and, hence, symmetry is maintained. There exists, therefore, a real polynomial approximation for u

$$u=[u]_0+\sum_{k=1}^{j}[u]_k\left(\frac{r-r_{0j}}{r_{jj}-r_{0j}}\right)^{(2k-1)} \tag{10}$$

where the $[u]_0$ and $[u]_k$ terms are functions of u at the strip boundaries. For $j=1$

$$u=[u]_0+[u]_1\left(\frac{r-r_{01}}{r_{11}-r_{01}}\right) \tag{11}$$

When $r=r_{01}$

$$[u]_0=u_{01} \tag{12}$$

and when $r=r_{11}$

$$[u]_1=u_{11}-u_{01} \tag{13}$$

The resulting equation for u is

$$u=u_{01}+(u_{11}-u_{01})\left(\frac{r-r_{01}}{r_{11}-r_{01}}\right) \tag{14}$$

Equation (14) is integrated between the limits of $r=r_{01}$ and $r=r_{11}$

$$\int_{r_{01}}^{r_{11}}u\,dr=\left(\frac{u_{01}+u_{11}}{2}\right)(r_{11}-r_{01}) \tag{15}$$

Substituting Eq. (15) into Eq. (9) transforms it into an ordinary differential equation which, when combined with the boundary conditions Eqs. (6) and (7), yields

$$\frac{dr_{01}}{dz}\frac{dw_{01}}{dz}+\frac{dr_{11}}{dz}\frac{dw_{11}}{dz}=\left(2+\frac{dr_{11}}{dz}\frac{dr_{01}}{dz}\right)\frac{w_{11}-w_{01}}{r_{11}-r_{01}}-w_{01}\frac{d^2r_{01}}{dz^2}-w_{11}\frac{d^2r_{11}}{dz^2}+\frac{w_{01}}{r_{11}-r_{01}}\left(\frac{dr_{11}}{dz}\right)^2-\frac{w_{01}}{r_{11}-r_{01}}\left(\frac{dr_{01}}{dz}\right)^2 \tag{16}$$

Since the shape of the nozzle is specified, r_{01}, r_{11} and their derivatives are known. The only unknowns are the axial velocities at the inner and outer walls, w_{01} and w_{11}, and their derivatives.

Similar treatment of the continuity equation produces a second differential equation in dw_{01}/dz and dw_{11}/dz. The integrand to be approximated in this case is w or ρw. Symmetry dictates that it must be an even powered polynomial and is, therefore, either

$$\rho w=\sum_{k=0}^{i}[\rho w]_k\left(\frac{r-r_{0j}}{r_{jj}-r_{0j}}\right)^{2k} \tag{17}$$

for compressible flow or

$$w=\sum_{k=0}^{i}[w]_k\left(\frac{r-r_{0j}}{r_{jj}-r_{0j}}\right)^{2k} \tag{18}$$

for incompressible flow.

Combining the two differential equations in dw_{01}/dz and dw_{11}/dz yields two new equations, one for dw_{01}/dz and the other for dw_{11}/dz. These equations may now be integrated by standard numerical methods.

Computational Considerations

To start the integration, initial values of the integrands, w_{01} and w_{11}, must be specified. These values may be specified directly or may be calculated from other specified parameters. It has been found that specifying the value of the ratio of w_{01} to w_{11} and the total flow rate is easier than specifying w_{01} and w_{11} individually. The ratio w_{01}/w_{11} is primarily a function of nozzle geometry. It is independent of the flow rate in incompressible flow. The total flow rate through the nozzle can be found by integrating the continuity equation between the inner and outer walls. The result for one strip is

$$\left[\left(8-5n+\frac{8nr_{01}}{r_{11}-r_{01}}\right)w_{01}+\left(4-n+\frac{4nr_{01}}{r_{11}-r_{01}}\right)w_{11}\right]\times(r_{11}-r_{01})^{(n+1)}=F_{\text{inc}} \tag{19}$$

for incompressible flow and is

$$\left[\left(8-5n+\frac{8nr_{01}}{r_{11}-r_{01}}\right)\rho_{01}w_{01}+\left(4-n+\frac{4nr_{01}}{r_{11}-r_{01}}\right)\rho_{11}w_{11}\right]\times(r_{11}-r_{01})^{(n+1)}=F_{\text{comp}} \tag{20}$$

for compressible flow.

In some problems, the flow rate is determined by conditions outside of the nozzle and would be known. In this case, the ratio of w_{01}/w_{11} is the only unknown. If the flow rate is determined by conditions inside the nozzle, then the flow rate must be iterated on along with w_{01}/w_{11}. In this instance, the advantage of using the ratio of w_{01}/w_{11} and the total flow rate as the specified parameters for starting the integration is that they are nearly independent of each other. Thus, the iteration required to establish the initial values of the integrands for a satisfactory solution of the flowfield can be performed on one parameter at a time. If the integrands themselves were used, the iteration would have to be done on both simultaneously, since they are mutually dependent.

What constitutes a satisfactory solution depends on the particular problem. In the ordinary nozzle[5-8] it was found that a saddle type singularity exists at the throat of the nozzle. In supersonic flow, a second singularity occurred at the intersection of the sonic line and the nozzle centerline. The effect on the integration is that if the initial values of the integrands are not correct, they diverge as the singularity is approached. One going to zero and the other increasing sharply. As better initial values of the integrands are used, the integration can approach the singularity more closely before the divergence becomes serious. It is impractical to reach the singularity itself, since the initial values of integrands would have to be correct to an infinite number of significant digits. Generally, the integration is stopped short of the singularity and is extrapolated up to it. The annular nozzle behaves the same as the ordinary nozzle, except that the singularity is not necessarily at the throat or at the sonic line.

The solution for the turbine blade passage has quite different characteristics. There is a singularity in this case also, but it is neither at the throat nor at the sonic line. Further, the singularity is a point type rather than the saddle type. All solutions, regardless of initial value, are the same at the singularity. In this case, the method used to determine the correct solution for ordinary and annular nozzles cannot be used. However, the directions and velocities of the flow approaching and leaving the turbine blade row are known and, hence, the force exerted on the blade row can be calculated from the velocity diagram. The pressure on the blades is integrated along with the velocity. The total force thus determined must equal the force derived from the velocity diagram. Different initial values of the integrands correspond physically to different directions and/or velocities approaching and leaving the blade row.

Higher Order Solutions

Higher order approximations can be made in the same manner, thereby improving the accuracy of the method, by increasing the number of strips used and, hence, the number of terms in the approximations. Advancing from one to two strips, the flowfield is divided by a line located midway between the inner and outer walls $[r_{12} = (r_{02}+r_{22})/2]$. The continuity and irrotationality equations are now integrated first from $r = r_{02}$ to $r = r_{12}$ and then from $r = r_{02}$ to $r = r_{22}$. This yields four ordinary differential equations in the unknowns, w_{02}, w_{12}, w_{22}, and u_{12}.

The four equations can be integrated using any standard method, as for the one strip case, but now four initial values must be specified. As with the one strip case, instead of specifying the initial value of the integrands directly, ratios of the integrands are used. The ratios are w_{02}/w_{22}, w_{12}/w_{22} and u_{12}/w_{12}. The fourth parameter is the total flow rate obtained by integrating the continuity equation between the inner and outer walls. The iterative procedure is to assume values for the ratios and the flow rate and to iterate on them until a satisfactory solution is obtained. The definition of a satisfactory solution depends heavily on the particular type of nozzle and various boundary conditions. For example, in the annular nozzle the inlet flow may be assumed to be uniform, in which case the initial value of u_{12} is zero. In some cases, such as subsonic flow in a turbine blade passage, the flow rate is determined by conditions outside the nozzle and hence the flow rate is known. During integration in the annular nozzle either w_{02} or w_{22} may go to zero at some value of z. In this case, it is a matter of iterating on the parameters until the integration can be carried up to the singularity without w_{02} or w_{22} going to zero and with the velocities being continuous across the singularity. In every case, there must be enough conditions known to isolate the appropriate solution from the infinite number of possible solutions. Solutions for three or more strips present many additional difficulties in the numerical analysis.

When either the inner or outer wall profile has curvature discontinuities, acceleration discontinuities arise in the solution. In incompressible flow and in subsonic compressible flow, these discontinuities are generally small, so the use of a small integration step size prevents them from perturbing the solution seriously. In supersonic flow, they can cause the integration to become unstable. An acceleration discontinuity sufficiently large to cause serious perturbations in the solution generally has been indicative of a velocity change in the real flow large enough to cause separation or shock waves. In either case, the solution is invalid since it assumes that the flow follows the wall and is shock free. Although the solution is invalid, it is useful in predicting conditions which could cause separation or shock waves in the real flowfield.

It has been found in all of the nozzles analyzed that the direction of integration is important. Integration towards the singularity is stable, while integration away from it is unstable. It has not been proven that all nozzles contain a singularity,

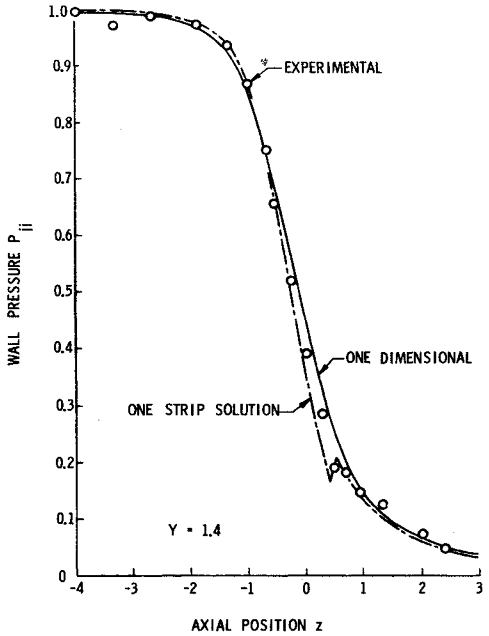

Fig. 2 Wall pressures for compressible flow in annular nozzles.

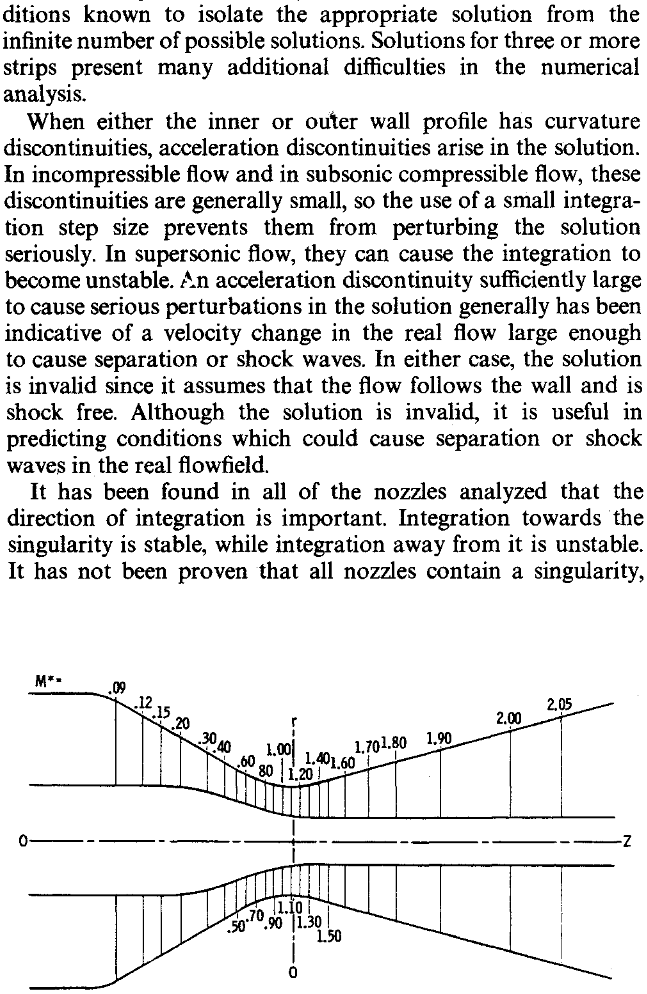

Fig. 3 Velocity field predicted by one-dimensional case for compressible flow in conic annular nozzle.

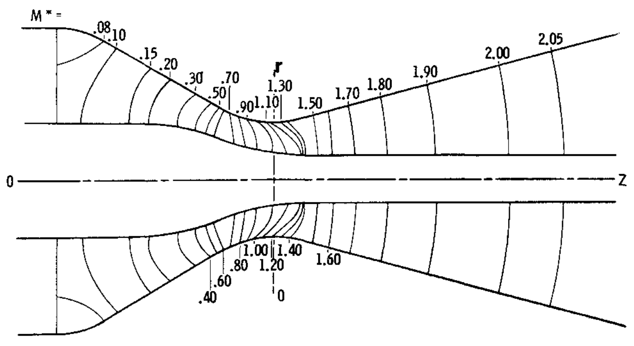

Fig. 4 Velocity field predicted by one-strip solution for compressible flow in annular nozzle.

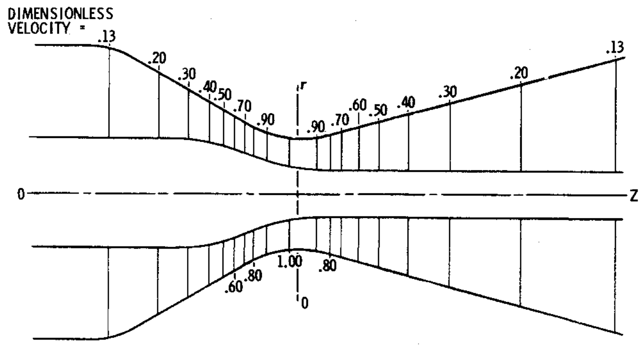

Fig. 6 Velocity field predicted by one-dimensional case for incompressible flow in conic annular nozzle.

but no nozzle has yet been found which does not contain at least one.

A basic nondimensional integration step size of 0.001 to 0.005 has been found to give reasonable computing time and accuracy. The computation time of a single iteration run for the annular nozzle was 90 sec on an IBM 360 model 50 and 48 sec on an IBM 370 model 165. For the asymmetric nozzle, the computation time using an IBM 370 model 165 was 55 sec.

Annular Nozzle Results

The annular nozzle was tested with air at ambient stagnation temperature in the School of Mechanical and Industrial Engineering at the University of New South Wales. Details of the experimental test setup are given in Ref. 7. The same nozzle was also tested with water in the Hydraulics Laboratory of the University. Experimental details are given in Ref. 6.

The inlet Reynolds numbers for the tests with air were from 5×10^5 to 1×10^6 based on the inlet diameter of the outer wall. The inlet Reynolds numbers with water were from 3×10^5 to 5×10^5. The inlet diameter of the nozzle used in this study was 4.06 in. The calculated one-strip solution and the experimental wall pressures for the compressible flow case are presented in Fig. 2, together with a one-dimensional flow solution for reference. A specific heat ratio γ of 1.4 was used in the calculations. The experimental results and the one-strip solution show reasonably good agreement over most of the nozzle except in the region between $z = 1$ and $z = 2$ and for one point at $z = -3.33$. The disagreement between $z = 1$ and $z = 2$ is probably due to the presence of a shock wave in the flowfield. The same nozzle without the center body had a shock wave which appeared to originate at the tangent point between the circular arc throat and the conic nozzle. The disagreement at $z = -3.33$ is probably due to a disturbance of the flow by the nose of the center body located about one and one-half inches upstream. No disagreement existed when the nozzle was run without the center body.

The predicted flowfield for one-dimensional flow is shown in Fig. 3 and for the one-strip solution using the method of integral relations in Fig. 4. For the incompressible case, the experimental and calculated wall pressures are presented in Fig. 5. The experimental results and the one-strip solution show good agreement down to the throat. Separation occurred at the throat and as a result, the measured pressures are considerably below the predicted values, which were based on unseparated flow. The one-strip solution is considerably more accurate than is the one-dimensional solution.

The wall pressure in Fig. 5 has been normalized to the flow conditions where the wall pressure will be zero where the velocity is a maximum. This method of normalizing the pressures forces both the one-dimensional flow solution and the one-strip solution to zero at the same point. The flow rate through the nozzle is not the same for both cases. If both flow rates were the same, then there would be a considerable difference between the two theoretical solutions, just as there is at $z = -1$ to $z = -2$. The flowfield for the one-dimensional solution is presented in Fig. 6 and for the one-strip solution in Fig. 7.

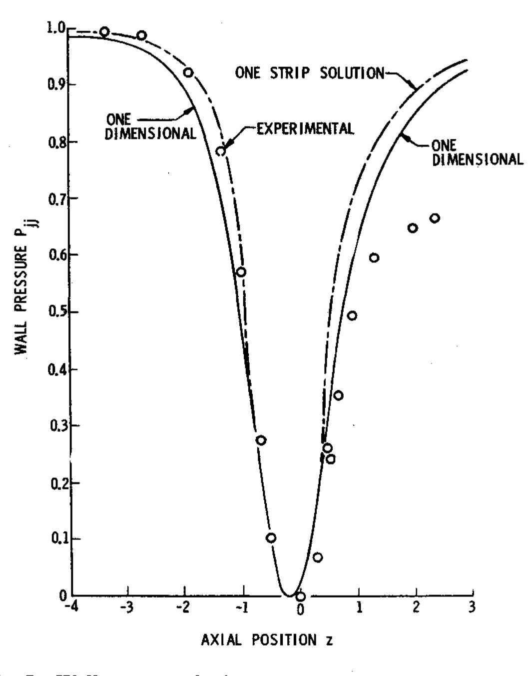

Fig. 5 Wall pressures for incompressible flow in annular nozzle.

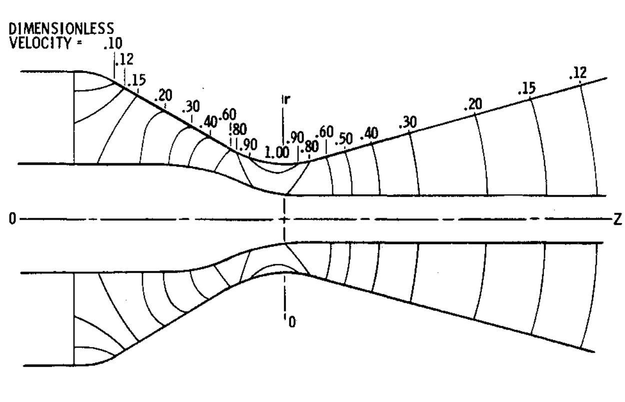

Fig. 7 Velocity field as predicted by one-strip solution for incompressible flow in annular nozzle.

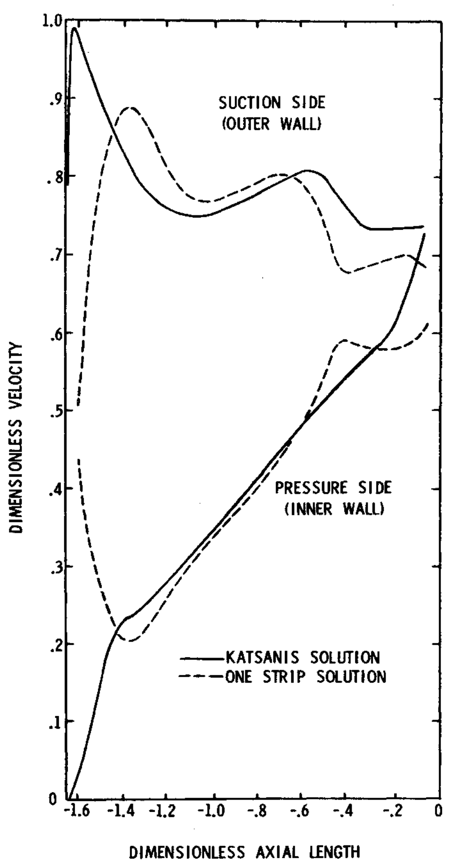

Fig. 8 Wall velocities for incompressible flow in asymmetric plane nozzle.

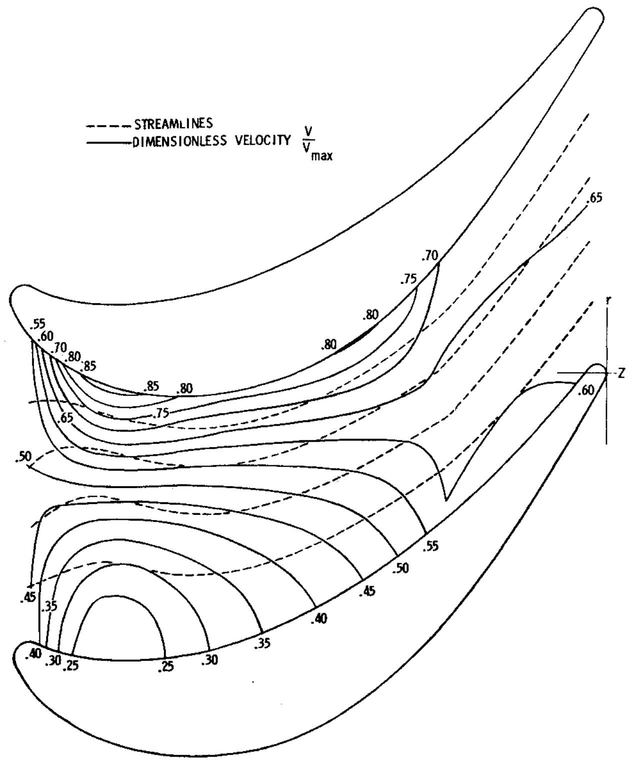

Fig. 9 Velocity field predicted by one-strip solution for incompressible flow in asymmetric plane nozzle.

The results of this study indicate that the one-strip solution using the method of integral relations shows good agreement with experimental results in an annular nozzle for both incompressible and compressible flow provided separation and shock waves are not present. The one-strip solution is considerably better in this case than is the one-dimensional solution.

Asymmetric Plane Nozzle

The particular example of an asymmetric plane nozzle used in this study is the passage between two blades of a mixed impulse-reaction turbine. In this case, the blade profile consists of a combination of mathematically defined curves so that not only is the shape of the passage rigorously defined but so are its first and second derivatives. The profile and its first derivative is continuous, but there are several discontinuities in the second derivative.

Experimental data are not available for this nozzle. However, it has been analyzed using a solution developed by Katsanis[9] for incompressible flow. Results from the one-strip solution using the method of integral relations is compared to the results of using the Katsanis method in Fig. 8. The two methods show similar results in the middle of the passage, but are markedly different at the inlet and at the exit. This disagreement arises from differences in the manner used to specify initial values. Because of inherent differences in the way the solutions are computed, it is impossible to specify identical initial conditions for the two methods. Which of the two solutions better predicts the actual conditions in the blade passage can only be determined experimentally. The predicted flowfield using the one-strip solution is presented in Fig. 9 for the incompressible flow case.

In both Figs. 8 and 9, the flow on the upper or suction side of the blade can be seen to first accelerate, then to decelerate, reaccelerate, redecelerate, reaccelerate and finally decelerate again as the trailing edge of the blade is approached. On the lower or pressure side, the velocity is initially decelerated, then increases until $z = -0.4$, where it decelerates again slightly and

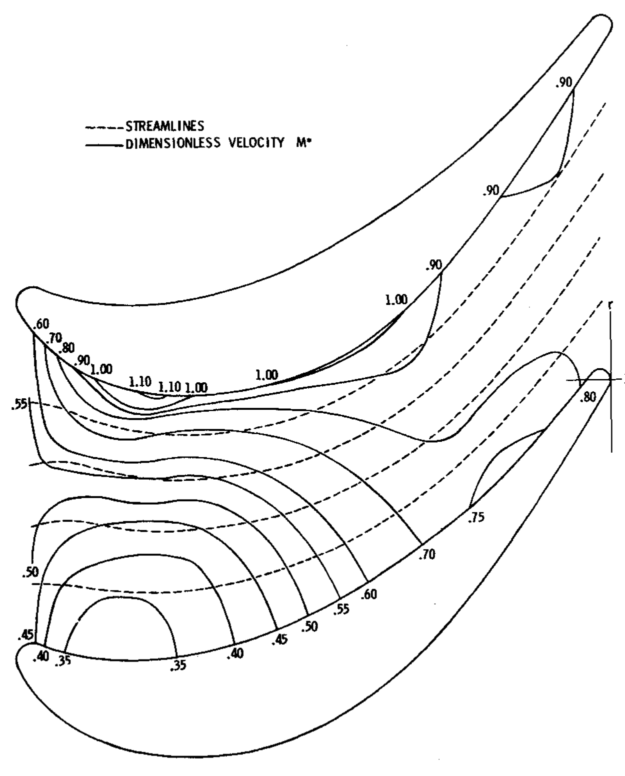

Fig. 10 Velocity field predicted by one-strip solution for compressible flow in asymmetric plane nozzle.

finally reaccelerates near the trailing edge. The sharp change in velocity on both the pressure and suction surfaces at $z = -0.4$ is the result of a discontinuity in the second derivative of the profile of the suction surface.

The compressible flow solution is shown in Fig. 10. Quantitatively, it is different from the incompressible flow solution of Fig. 9, but qualitatively they are very similar. Both show the same three velocity peaks on the suction surface of the blade and the peak on the pressure surface at $z = -0.4$. Both indicate that the streamlines are hooked at the blade inlet, although it is less evident in the compressible flow solution than in the incompressible flow solution. Since the approach angle of the flow α (Fig. 1) is 38°, some of the streamlines near the pressure surface of the blade must follow an S-shaped path at the passage entrance. The S-shaped turn is the result of the effect of the leading edge of the blade and of the increased streamtube widths near the lower (pressure) surface, where the flow velocity is low.

As the flow travels through the blade passage, it becomes more uniform but even at the trailing edge it still shows some nonuniformity. The degree of nonuniformity is about the same for both compressible and incompressible cases.

Conclusions

In this study, only the one-strip solution was used. It greater accuracy is required, two or more strips may be used. In addition, it is possible, by changing the basic equations, to include real gas effects and nonisentropic flows.

There are limitations to the use of the method. One of these is the assumption that the velocity components are continuous across the strips. This requires that there be no shocks in the flow. The second is that the approximations used may not be adequate. In nozzles with very small radii of curvature, the change in velocity may be so great that a high order approximation may be required to achieve the desired accuracy. This would necessitate a large number of strips, and hence considerable computing time and program complexity. Depending on the boundary conditions prescribed, flow separation may also invalidate the solution. These limitations are not unique to this method, but are shared by many computational methods used for fluid flow problems.

The method of integral relations has been used successfully to predict the flowfield in annular and asymmetric plane nozzles. The predicted fields are in reasonable agreement with experimental results in the former case, and with results predicted by another computational method in the latter.

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